

WEAKLY Z SYMMETRIC MANIFOLDS

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ABSTRACT. We introduce a new kind of Riemannian manifold that includes weakly-, pseudo- and pseudo projective- Ricci symmetric manifolds. The manifold is defined through a generalization of the so called Z tensor; it is named *weakly Z symmetric* and denoted by $(WZS)_n$. If the Z tensor is singular we give conditions for the existence of a proper concircular vector. For non singular Z tensor, we study the closedness property of the associated covectors and give sufficient conditions for the existence of a proper concircular vector in the conformally harmonic case, and the general form of the Ricci tensor. For conformally flat $(WZS)_n$ manifolds, we derive the local form of the metric tensor.

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1. INTRODUCTION

In 1993 Tamassy and Binh [31] introduced and studied a Riemannian manifold whose Ricci tensor¹ satisfies the equation:

$$(1) \quad \nabla_k R_{jl} = A_k R_{jl} + B_j R_{kl} + D_l R_{kj}.$$

The manifold is called *weakly Ricci symmetric* and denoted by $(WRS)_n$. The covectors A_k , B_k and D_k are the *associated 1-forms*. The same manifold with the 1-form A_k replaced by $2A_k$ was studied by Chaki and Koley [6], and called *generalized pseudo Ricci symmetric*. The two structures extend *pseudo Ricci symmetric* manifolds, $(PRS)_n$, introduced by Chaki [4], where $\nabla_k R_{jl} = 2A_k R_{jl} + A_j R_{kl} + A_l R_{kj}$ (this definition differs from that of R. Deszcz [17]).

Later on, other authors studied the manifolds [10, 20, 12]; in [12] some global properties of $(WRS)_n$ were obtained, and the form of the Ricci tensor was found. In [10] generalized pseudo Ricci symmetric manifolds were considered, where the conformal curvature tensor

$$(2) \quad C_{jkl}{}^m = R_{jkl}{}^m + \frac{1}{n-2}(\delta_j{}^m R_{kl} - \delta_k{}^m R_{jl} + R_j{}^m g_{kl} - R_k{}^m g_{jl}) - \frac{R}{(n-1)(n-2)}(\delta_j{}^m g_{kl} - \delta_k{}^m g_{jl})$$

vanishes (for $n = 3$: $C_{jkl}{}^m = 0$ holds identically, [27]) and the existence of a proper concircular vector was proven. In [20] a quasi conformally flat $(WRS)_n$ was studied, and again the existence of a proper concircular vector was proven.

In [2] $(PRS)_n$ with harmonic curvature tensor (i.e. $\nabla_m R_{jkl}{}^m = 0$) or with harmonic conformal curvature tensor (i.e. $\nabla_m C_{jkl}{}^m = 0$) were considered.

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¹Here we define the Ricci tensor as $R_{kl} = -R_{mkl}{}^m$ and the scalar curvature as $R = g^{ij} R_{ij}$. ∇_k is the covariant derivative with reference to the metric g_{kl} . We also put $\|\eta\| = \sqrt{\eta^k \eta_k}$.

Chaki and Saha considered the projective Ricci tensor P_{kl} , obtained by a contraction of the projective curvature tensor $P_{jkl}{}^m$ [18]:

$$(3) \quad P_{kl} = \frac{n}{n-1} \left(R_{kl} - \frac{R}{n} g_{kl} \right),$$

and generalized $(PRS)_n$ to manifolds such that

$$(4) \quad \nabla_k P_{jl} = 2A_k P_{jl} + A_j P_{kl} + A_l P_{kj}.$$

The manifold is called *pseudo projective Ricci symmetric* and denoted by $(PWR S)_n$ [8]. Recently another generalization of a $(PRS)_n$ was considered in [5] and [11], whose Ricci tensor satisfies the condition

$$(5) \quad \nabla_k R_{jl} = (A_k + B_k) R_{jl} + A_j R_{kl} + A_l R_{kj},$$

The manifold is called *almost pseudo Ricci symmetric* and denoted by $A(PRS)_n$. In ref.[11] the properties of conformally flat $A(PRS)_n$ were studied, pointing out their importance in the theory of General Relativity.

It seems worthwhile to introduce and study a new manifold structure that includes $(WRS)_n$, $(PRS)_n$ and $(PWR S)_n$ as special cases.

Definition 1.1. A $(0,2)$ symmetric tensor is a *generalized Z tensor* if

$$(6) \quad Z_{kl} = R_{kl} + \phi g_{kl},$$

where ϕ is an arbitrary scalar function. The Z scalar is $Z = g^{kl} Z_{kl} = R + n\phi$.

The classical Z tensor is obtained with the choice $\phi = -\frac{1}{n}R$. Hereafter we refer to the generalized Z tensor simply as the Z tensor.

The Z tensor allows us to reinterpret several well known structures on Riemannian manifolds.

- 1) If $Z_{kl} = 0$ the (Z -flat) manifold is an Einstein space, $R_{ij} = (R/n)g_{ij}$ [3].
- 2) If $\nabla_i Z_{kl} = \lambda_i Z_{kl}$, the (Z -recurrent) manifold is a generalized Ricci recurrent manifold [9, 26]: the condition is equivalent to $\nabla_i R_{kl} = \lambda_i R_{kl} + (n-1)\mu_i g_{kl}$ where $(n-1)\mu_i \equiv (\lambda_i - \nabla_i)\phi$. If moreover $0 = (\lambda_i - \nabla_i)\phi$, a Ricci Recurrent manifold is recovered.
- 3) If $\nabla_k Z_{jl} = \nabla_j Z_{kl}$ (i.e. Z is a Codazzi tensor, [16]) then $\nabla_k R_{jl} - \nabla_j R_{kl} = (g_{kl}\nabla_j - g_{jl}\nabla_k)\phi$. By transvecting with g^{jl} we get $\nabla_k[R + 2(n-1)\phi] = 0$ and, finally,

$$\nabla_k R_{jl} - \nabla_j R_{kl} = \frac{1}{2(n-1)}(g_{jl}\nabla_k - g_{kl}\nabla_j)R.$$

This condition defines a *nearly conformally symmetric* manifold, $(NCS)_n$. The condition was introduced and studied by Roter [29]. Conversely a $(NCS)_n$ has a Codazzi Z tensor if $\nabla_k[R + 2(n-1)\phi] = 0$.

- 4) Einstein's equations [14] with cosmological constant Λ and energy-stress tensor T_{kl} may be written as $Z_{kl} = kT_{kl}$, where $\phi = -\frac{1}{2}R + \Lambda$, and k is the gravitational constant. The Z tensor may be thought of as a generalized Einstein gravitational tensor with arbitrary scalar function ϕ .

Conditions on the energy-momentum tensor determine constraints on the Z tensor: the vacuum solution $Z = 0$ determines an Einstein space with $\Lambda = \frac{n-2}{2n}R$; conservation of total energy-momentum ($\nabla^l T_{kl} = 0$) gives $\nabla^l Z_{kl} = 0$ and $\nabla_k(\frac{1}{2}R + \phi) = 0$; the condition $\nabla_i Z_{kl} = 0$ describes a space-time with conserved energy-momentum

density.

Several cases accomodate in a new kind of Riemannian manifold:

Definition 1.2. A manifold is *Weakly Z symmetric*, and denoted by $(WZS)_n$, if the generalized Z tensor satisfies the condition:

$$(7) \quad \nabla_k Z_{jl} = A_k Z_{jl} + B_j Z_{kl} + D_l Z_{kj}.$$

If $\phi = 0$ we recover a $(WRS)_n$ and its particular case $(PRS)_n$. If $\phi = -R/n$ (classical Z tensor) and if A_k is replaced by $2A_k$, $B_k = D_k = A_k$, then $Z_{jl} = \frac{n-1}{n}P_{jl}$ and the space reduces to a $(PWRs)_n$.

In sect.2 we obtain general properties of $(WZS)_n$ that descend directly from the definition and strongly depend on Z_{ij} being singular or not. The two cases are examined in sections 3 and 4. In sect.3 we study $(WZS)_n$ that are conformally or pseudo conformally harmonic with $B - D \neq 0$; we show that $B - D$, after normalization, is a proper concircular vector. Sect.4 is devoted to $(WZS)_n$ with non-singular Z tensor, and gives conditions for the closedness of the 1-form $A - B$ that involve various generalized curvature tensors. In sect.5 we study conformally harmonic $(WZS)_n$ and obtain the explicit form of the Ricci tensor. In the conformally flat case we also give the local form of the metric.

2. GENERAL PROPERTIES

From the definition of a $(WZS)_n$ and its symmetries we obtain

$$(8) \quad 0 = \eta_j Z_{kl} - \eta_l Z_{kj},$$

$$(9) \quad \nabla_k Z_{jl} - \nabla_j Z_{kl} = \omega_k Z_{jl} - \omega_j Z_{kl},$$

with covectors $\omega_k = A_k - B_k$ and $\eta_k = B_k - D_k$ that will be used throughout. Let's consider eq.(8) first, it implies the following statements:

Proposition 2.1. *In a $(WZS)_n$, if the Z tensor is non-singular then $\eta_k = 0$.*

Proof. If the Z tensor is non singular, there exists a $(2,0)$ tensor Z^{-1} such that $(Z^{-1})^{kh} Z_{kl} = \delta^h_l$. By transvecting eq.(8) with $(Z^{-1})^{kh}$ we obtain $\eta_j \delta_l^h = \eta_l \delta_j^h$; put $h = l$ and sum to obtain $(n-1)\eta_j = 0$. \square

Proposition 2.2. *If $\eta_k \neq 0$ and the scalar $Z \neq 0$, then the Z tensor has rank one:*

$$(10) \quad Z_{ij} = Z \frac{\eta_i \eta_j}{\eta^k \eta_k}$$

Proof. Multiply eq.(8) by η^j and sum: $\eta^j \eta_j Z_{kl} = \eta_l \eta^j Z_{kj}$. Multiply eq.(8) by g^{jk} and sum: $\eta^k Z_{kl} = Z \eta_l$. The two results imply the assertion. \square

The result translates to the Ricci tensor, whose expression is characteristic of *quasi Einstein* Riemannian manifolds [7], and generalizes the results of [12]:

Proposition 2.3. *A $(WZS)_n$ with $\eta_k \neq 0$, is a quasi Einstein manifold:*

$$(11) \quad R_{ij} = -\phi g_{ij} + Z T_i T_j, \quad T_i = \frac{\eta_i}{\|\eta\|},$$

Next we consider eq.(9). If Z_{ij} is a Codazzi tensor, then the l.h.s. of the equation vanishes by definition, and the above discussion of eq.(8) can be repeated. We merely state the result:

Proposition 2.4. *In a $(WZS)_n$ with a Codazzi Z tensor, if Z is singular then $\omega_k \neq 0$. Conversely, if $\text{rank}[Z_{kl}] > 1$ then $\omega_k = 0$.*

3. HARMONIC CONFORMAL OR QUASI CONFORMAL $(WZS)_n$ WITH $\eta \neq 0$

In this section we consider manifolds $(WZS)_n$ ($n > 3$) with $\eta_k \neq 0$, and the property $\nabla_m C_{jkl}{}^m = 0$ (i.e. harmonic conformal curvature tensor [3]) or $\nabla_m W_{jkl}{}^m = 0$ (i.e. harmonic quasi conformal curvature tensor [34]). We provide sufficient conditions for $\eta/\|\eta\|$ to be a proper concircular vector [28, 32].

We begin with the case of harmonic conformal tensor. From the expression for the divergence of the conformal tensor,

$$(12) \quad \nabla_m C_{jkl}{}^m = \frac{n-3}{n-2} \left[\nabla_k R_{jl} - \nabla_j R_{kl} + \frac{1}{2(n-1)} (g_{kl} \nabla_j - g_{jl} \nabla_k) R \right]$$

we read the condition $\nabla_m C_{jkl}{}^m = 0$:

$$(13) \quad \nabla_k R_{jl} - \nabla_j R_{kl} = \frac{1}{2(n-1)} (g_{jl} \nabla_k - g_{kl} \nabla_j) R.$$

We need the following theorem, whose proof given here is different from that in [13] (see also [10]):

Theorem 3.1. *Let M be a $n > 3$ dimensional manifold, with harmonic conformal curvature tensor, and Ricci tensor $R_{kl} = \alpha g_{kl} + \beta T_k T_l$, where α, β are scalars, and $T^k T_k = 1$. If*

$$(14) \quad (T_j \nabla_k - T_k \nabla_j) \beta = 0,$$

then T_k is a proper concircular vector.

Proof. Since M is conformally harmonic, eq.(13) gives:

$$(15) \quad \beta [\nabla_k (T_j T_l) - \nabla_j (T_k T_l)] = \frac{1}{2(n-1)} (g_{jl} \nabla_k - g_{kl} \nabla_j) S,$$

where $S = -(n-2)\alpha + \beta$, and condition (14) was used. The proof is in four steps.
1) We show that $T^l \nabla_l T_k = 0$: multiply eq.(15) by g^{jl} to obtain: a) $-\beta \nabla^l (T_k T_l) = \frac{1}{2} \nabla_k S$. The result a) is multiplied by T^k to give: b) $-\beta \nabla_l T^l = \frac{1}{2} T^l \nabla_l S$. a) and b) combine to give: c) $-\beta T^l \nabla_l T_k = \frac{1}{2} [\nabla_k - T_k T^l \nabla_l] S$. Finally multiply eq.(15) by $T^k T^l$ and use the property $T^l \nabla_k T_l = 0$ to obtain:

$$\beta T^k \nabla_k T_j = \frac{1}{2(n-1)} (T_j T^k \nabla_k - \nabla_j) S$$

which, compared to c) shows that d) $T^l \nabla_l T_k = 0$ and $(T_j T^k \nabla_k - \nabla_j) S = 0$.

2) We show that T is a closed 1-form: multiply eq.(15) by T^l

$$\beta [\nabla_k T_j - \nabla_j T_k] = \frac{1}{2(n-1)} (T_j \nabla_k - T_k \nabla_j) S.$$

T is a closed form if the r.h.s. is null. This is proven by using identity a) to write: $(T_j \nabla_k - T_k \nabla_j) S = -2\beta [T_j \nabla^l (T_k T_l) - T_k \nabla^l (T_j T_l)] = 0$ by property d).

3) With condition d) in mind, transvect eq.(15) with T^k and obtain

$$-\beta \nabla_j T_l = \frac{1}{2(n-1)} (g_{jl} T^k \nabla_k - T_l \nabla_j) S$$

Use d) to replace $T_l \nabla_j S$ with $T_l T_j T^k \nabla_k S$. Then:

$$(16) \quad \nabla_j T_l = f (T_j T_l - g_{jl}), \quad f \equiv \frac{T^k \nabla_k S}{2\beta(n-1)}$$

which means that T_k is a concircular vector.

4) We prove that T_k is a proper concircular vector, i.e. fT_k is a closed 1-form: from d) by a covariant derivative we obtain $\nabla_j \nabla_k S = (\nabla_j T_k)(T^l \nabla_l S) + T_k \nabla_j (T^l \nabla_l S)$; subtract same equation with indices k and j exchanged. Since T_k is a closed 1-form we obtain: $T_k \nabla_j (T^l \nabla_l S) = T_j \nabla_k (T^l \nabla_l S)$. Multiply by T^k :

$$(T_j T^k \nabla_k - \nabla_j)(T^l \nabla_l S) = 0$$

From the relation (14), one obtains: $(T_k T^l \nabla_l - \nabla_k)\beta = 0$. It follows that the scalar function f has the property $\nabla_j f = \mu T_j$ where μ is a scalar function. Then the 1-form fT_k is closed. \square

With the identifications $\alpha = -\phi$ and $\beta = Z$, $T_i = \eta_i / \|\eta\|$ (see Prop. 2.3) the condition (14) is $(\eta_j \nabla_k - \eta_k \nabla_j)Z = 0$. Since $Z = S - (n-2)\phi$ and $(\eta_j \nabla_k - \eta_k \nabla_j)S = 0$, the condition can be rewritten as $(\eta_j \nabla_k - \eta_k \nabla_j)\phi = 0$. Thus we can state the following:

Theorem 3.2. *In a $(WZS)_n$ manifold with $\eta_k \neq 0$ and harmonic conformal curvature tensor, if*

$$(17) \quad (\eta_j \nabla_k - \eta_k \nabla_j)\phi = 0$$

then $\eta_i / \|\eta\|$ is a proper concircular vector.

Remark 1. *If $\phi = 0$ or $\nabla_k \phi = 0$, the condition (17) is fulfilled automatically. In the case $\phi = 0$ we recover a $(WRS)_n$ manifold (and the results of refs [10, 12]).*

Now we consider the case of a $(WZS)_n$ manifold with harmonic quasi conformal curvature tensor. In 1968 Yano and Sawaki [34] defined and studied a tensor W_{jkl}^m on a Riemannian manifold of dimension $n > 3$, which includes as particular cases the conformal curvature tensor C_{jkl}^m , eq.(2), and the concircular curvature tensor

$$(18) \quad \tilde{C}_{jkl}^m = R_{jkl}^m + \frac{R}{n(n-1)}(\delta_j^m g_{kl} - \delta_k^m g_{jl}).$$

The tensor is known as the *quasi conformal* curvature tensor:

$$(19) \quad W_{jkl}^m = -(n-2)b C_{jkl}^m + [a + (n-2)b]\tilde{C}_{jkl}^m;$$

a and b are nonzero constants. From the expressions (12) and (32) we evaluate

$$(20) \quad \nabla_m W_{jkl}^m = (a+b)\nabla_m R_{jkl}^m + \frac{2a - b(n-1)(n-4)}{2n(n-1)}(g_{kl}\nabla_j - g_{jl}\nabla_k)R.$$

A manifold is *quasi conformally harmonic* if $\nabla_m W_{jkl}^m = 0$. By transvecting the condition with g^{jk} we get:

$$(21) \quad (1 - 2/n)[a + b(n-2)] \nabla_j R = 0,$$

which means that either $a + b(n-2) = 0$ or $\nabla_j R = 0$. The first condition implies $W = C$, and gives back the harmonic conformal case. If $\nabla_j R = 0$ it is $\nabla_m R_{jkl}^m = 0$ by (20), and the equations in the proof of theorem 3.1 simplify and we can state the following (analogous to theorem 3.2):

Theorem 3.3. *Let $(WZS)_n$ be a quasi conformally harmonic manifold, with $\eta_k \neq 0$. If $(\eta_j \nabla_k - \eta_k \nabla_j)\phi = 0$, then $\eta/\|\eta\|$ is a proper concircular vector.*

4. $(WZS)_n$ WITH NON-SINGULAR Z TENSOR: CONDITIONS FOR CLOSED ω

In this section we investigate in a $(WZS)_n$ ($n > 3$) the conditions the 1-form ω_k to be closed: $\nabla_i \omega_j - \nabla_j \omega_i = 0$. We need:

Lemma 4.1 (Lovelock's differential identity, [23, 24]). *In a Riemannian manifold the following identity is true:*

$$(22) \quad \begin{aligned} & \nabla_i \nabla_m R_{jkl}{}^m + \nabla_j \nabla_m R_{kil}{}^m + \nabla_k \nabla_m R_{ijl}{}^m \\ & = -R_{im} R_{jkl}{}^m - R_{jm} R_{kil}{}^m - R_{km} R_{ijl}{}^m \end{aligned}$$

and also the contracted second Bianchi identity in the form

$$(23) \quad \nabla_m R_{jkl}{}^m = \nabla_k Z_{jl} - \nabla_j Z_{kl} + (g_{kl} \nabla_j - g_{jl} \nabla_k) \phi.$$

Now we prove the relevant theorem (see also [24]):

Theorem 4.2. *In a $(WZS)_n$ ($n > 3$) with non singular Z tensor, ω_k is a closed 1-form if and only if:*

$$(24) \quad R_{im} R_{jkl}{}^m + R_{jm} R_{kil}{}^m + R_{km} R_{ijl}{}^m = 0.$$

Proof. The covariant derivative of eq.(23) and eq.(9) give: $\nabla_i \nabla_m R_{jkl}{}^m = (\nabla_i \omega_k) Z_{jl} + \omega_k (\nabla_i Z_{jl}) - (\nabla_i \omega_j) Z_{kl} - \omega_j (\nabla_i Z_{kl}) + (g_{kl} \nabla_i \nabla_j \phi - g_{jl} \nabla_i \nabla_k \phi)$. Cyclic permutations of the indices i, j, k are made, and the resulting three equations are added:

$$\begin{aligned} & \nabla_i \nabla_m R_{jkl}{}^m + \nabla_j \nabla_m R_{kil}{}^m + \nabla_k \nabla_m R_{ijl}{}^m \\ & = (\nabla_i \omega_k - \nabla_k \omega_i) Z_{jl} + (\nabla_j \omega_i - \nabla_i \omega_j) Z_{kl} + (\nabla_k \omega_j - \nabla_j \omega_k) Z_{il} \\ & \quad + \omega_j (\nabla_k Z_{il} - \nabla_i Z_{kl}) + \omega_k (\nabla_i Z_{jl} - \nabla_j Z_{il}) + \omega_i (\nabla_j Z_{kl} - \nabla_k Z_{jl}). \end{aligned}$$

Cancellations occur by eq.(9). By lemma 4.1, one obtains:

$$\begin{aligned} & -R_{im} R_{jkl}{}^m - R_{jm} R_{kil}{}^m - R_{km} R_{ijl}{}^m \\ & = (\nabla_i \omega_k - \nabla_k \omega_i) Z_{jl} + (\nabla_j \omega_i - \nabla_i \omega_j) Z_{kl} + (\nabla_k \omega_j - \nabla_j \omega_k) Z_{il}. \end{aligned}$$

If ω_k is a closed 1-form then eq.(24) is fulfilled. Conversely, suppose that eq.(24) holds: if the Z tensor is non singular, there is a $(2, 0)$ tensor such that $Z_{kl}(Z^{-1})^{km} = \delta_l^m$. Multiply the last equation by $(Z^{-1})^{hl}$: $(\nabla_i \omega_k - \nabla_k \omega_i) \delta_j^h + (\nabla_j \omega_i - \nabla_i \omega_j) \delta_k^h + (\nabla_k \omega_j - \nabla_j \omega_k) \delta_i^h = 0$. Set $h = j$ and sum: $(n-2)(\nabla_i \omega_k - \nabla_k \omega_i) = 0$. Since $n > 2$, ω_k is a closed 1-form. \square

Remark 2. *By Lovelock's identity, the condition (24) is obviously true if $\nabla_m R_{ijk}{}^m = 0$, i.e. the $(WZS)_n$ is a harmonic manifold. However, we have shown in ref.[24] that there is a broad class of generalized curvature tensors for which the case $\nabla_m K_{ijk}{}^m = 0$ implies the same condition. This class includes several well known curvature tensors, and is the main subject of this section.*

Definition 4.3. A tensor $K_{jkl}{}^m$ is a generalized curvature tensor² if:

- 1) $K_{jkl}{}^m = -K_{kjl}{}^m$,
- 2) $K_{jkl}{}^m + K_{klj}{}^m + K_{ljk}{}^m = 0$.

²The notion was introduced by Kobayashi and Nomizu [22], but with the further antisymmetry in the last pair of indices.

The second Bianchi identity does not hold in general, and is modified by a tensor source $B_{ijkl}{}^m$ that depends on the specific form of the curvature tensor:

$$(25) \quad \nabla_i K_{jkl}{}^m + \nabla_j K_{kil}{}^m + \nabla_k K_{ijl}{}^m = B_{ijkl}{}^m$$

Proposition 4.4 ([24]). *If $K_{jkl}{}^m$ is a generalized curvature tensor such that*

$$(26) \quad \nabla_m K_{jkl}{}^m = A \nabla_m R_{jkl}{}^m + B(a_{lk} \nabla_j - a_{lj} \nabla_k) \psi,$$

where $A \neq 0$, B are constants, ψ is a scalar field, and a_{ij} is a symmetric $(0,2)$ Codazzi tensor (i.e. $\nabla_i a_{kl} = \nabla_k a_{il}$), then the following relation holds:

$$(27) \quad \begin{aligned} \nabla_i \nabla_m K_{jkl}{}^m + \nabla_j \nabla_m K_{kil}{}^m + \nabla_k \nabla_m K_{ijl}{}^m \\ = -A(R_{im} R_{jkl}{}^m + R_{jm} R_{kil}{}^m + R_{km} R_{ijl}{}^m). \end{aligned}$$

Remark 3. In [16] it is proven that any smooth manifold carries a metric such that (M, g) admits a non trivial Codazzi tensor (i.e. proportional to the metric tensor) and the deep consequences on the structure of the curvature operator are presented (see also [25]).

Given a Codazzi tensor it is possible to exhibit a K tensor that satisfies the condition (26):

$$(28) \quad K_{jkl}{}^m = A R_{jkl}{}^m + B \psi (\delta_j^m a_{kl} - \delta_k^m a_{jl}).$$

Its trace is: $K_{kl} = -K_{mkl}{}^m = A R_{kl} - B(n-1)\psi a_{kl}$. Note that for $a_{kl} = g_{kl}$ the tensor K_{kl} is up to a factor a Z tensor. Thus Z tensors arise naturally from the invariance of Lovelock's identity.

Remark 4. In the literature one meets generalized curvature tensors whose divergence has the form (26), with trivial Codazzi tensor:

$$(29) \quad \nabla_m K_{jkl}{}^m = A \nabla_m R_{jkl}{}^m + B(g_{kl} \nabla_j - g_{jl} \nabla_k) R.$$

They are the projective curvature tensor $P_{jkl}{}^m$ [18], the conformal curvature tensor $C_{jkl}{}^m$ [27], the concircular tensor $\tilde{C}_{jkl}{}^m$ [28, 32], the conharmonic tensor $N_{jkl}{}^m$ [26, 30] and the quasi conformal tensor $W_{jkl}{}^m$ [34].

Definition 4.5. A manifold is K -harmonic if $\nabla_m K_{jkl}{}^m = 0$.

Proposition 4.6. *In a K -harmonic manifold, if K is of type (29) and $A \neq 2(n-1)B$, then $\nabla_j R = 0$.*

Proof. By transvecting eq.(29) with g^{kl} and by the second contracted Bianchi identity, we obtain $\frac{1}{2}[A - 2(n-1)B]\nabla_j R = 0$. \square

Hereafter, we specialize to $(WZS)_n$ manifolds with non singular Z tensor, and with a generalized curvature tensor of the type (29). From eqs. (23) and (9) we obtain:

$$(30) \quad \nabla_m K_{jkl}{}^m = A(\omega_k Z_{jl} - \omega_j Z_{kl}) + (g_{kl} \nabla_j - g_{jl} \nabla_k)(A\phi + B R).$$

Then, the manifold is K -harmonic if:

$$(31) \quad A(\omega_k Z_{jl} - \omega_j Z_{kl}) = (g_{jl} \nabla_k - g_{kl} \nabla_j)(A\phi + B R).$$

Lemma 4.7. *In a K -harmonic $(WZS)_n$ with non singular Z tensor:*

- 1) $\omega_k = 0$ if and only if $\nabla_k(A\phi + BR) = 0$;
- 2) If $A \neq 2(n-1)B$, then $\omega_k = 0$ if and only if $\nabla_k \phi = 0$.

Proof. If $\nabla_k(A\phi + BR) = 0$ then $\omega_k Z_{jl} = \omega_j Z_{kl}$: if the Z tensor is non singular, by transvecting with $(Z^{-1})^{lh}$ we obtain $\omega_j \delta^h_k = \omega_k \delta^h_j$. Now put $h = j$ and sum to obtain $(n-1)\omega_k = 0$. On the other hand if $\omega_k = 0$ eq.(31) gives $[g_{jl}\nabla_k - g_{kl}\nabla_j](A\phi + BR) = 0$ and transvecting with g^{kl} we get the result. If $A \neq 2B(n-1)$ then $\nabla_k R = 0$ and part 1) applies. \square

Theorem 4.8. *In a K -harmonic $(WZS)_n$ with non-singular Z tensor and K of type (29), if $\omega \neq 0$ then ω is a closed 1-form.*

This theorem extends theorem 4.2 (where $K = R$), and has interesting corollaries according to the various choices $K = C, W, P, \tilde{C}, N$.

Corollary 4.9. *Let $(WZS)_n$ have non singular Z tensor and $\omega \neq 0$. If $\nabla_m K_{jkl}{}^m = 0$, and $K = P, \tilde{C}, N$, then ω is a closed 1-form.*

Proof. 1) Harmonic conformal curvature: $\nabla_m C_{jkl}{}^m = 0$. Note that in this case $A = 2B(n-1)$; theorem 4.8 applies.

2) Harmonic quasi conformal curvature: $\nabla_m W_{jkl}{}^m = 0$: Eq.(21) gives either $\nabla_j R = 0$ or $a + b(n-2) = 0$. If $\nabla_j R = 0$ then $\nabla_m R_{jkl}{}^m = 0$ and theorem 4.2. If $a + b(n-2) = 0$ it is $\nabla_m C_{jkl}{}^m = 0$ and case 1) applies.

3) Harmonic projective curvature: $\nabla_m P_{jkl}{}^m = 0$. The components of the projective curvature tensor are [18, 30]:

$$P_{jkl}{}^m = R_{jkl}{}^m + \frac{1}{n-1}(\delta_j^m R_{kl} - \delta_k^m R_{jl}).$$

One evaluates $\nabla_m P_{jkl}{}^m = \frac{n-2}{n-1}\nabla_m R_{jkl}{}^m$, and theorem 4.2 applies.

4) Harmonic concircular curvature: $\nabla_m \tilde{C}_{jkl}{}^m = 0$. The concircular curvature tensor is given in eq.(18), [28, 32]. Its divergence is

$$(32) \quad \nabla_m \tilde{C}_{jkl}{}^m = \nabla_m R_{jkl}{}^m + \frac{1}{n(n-1)}(g_{kl}\nabla_j - g_{jl}\nabla_k)R$$

Theorem 4.8 applies.

5) Harmonic conharmonic curvature: $\nabla_m N_{jkl}{}^m = 0$. The conharmonic curvature tensor [26, 30] is:

$$N_{jkl}{}^m = R_{jkl}{}^m + \frac{1}{n-2}(\delta_j^m R_{kl} - \delta_k^m R_{jl} + R_j^m g_{kl} - R_k^m g_{jl}).$$

A covariant derivative and the second contracted Bianchi identity give:

$$\nabla_m N_{jkl}{}^m = \frac{n-3}{n-2}\nabla_m R_{jkl}{}^m + \frac{1}{2(n-2)}(g_{kl}\nabla_j - g_{jl}\nabla_k)R.$$

Theorems 4.8 applies. \square

There are other cases where the 1-form ω_k is closed for a $(WZS)_n$ manifold.

Definition 4.10 ([24, 21]). A n -dimensional Riemannian manifold is K -recurrent, $(KR)_n$, if the generalized curvature tensor is recurrent, $\nabla_i K_{jkl}{}^m = \lambda_i K_{jkl}{}^m$, for some non zero covector λ_i .

Theorem 4.11 ([24]). *In a $(KR)_n$, if λ_i is closed then:*

$$(33) \quad R_{im}R_{jkl}{}^m + R_{jm}R_{kil}{}^m + R_{km}R_{ijl}{}^m = -\frac{1}{A}\nabla_m B_{ijkl}{}^m.$$

where B is the source tensor in eq.(25). In particular, for $K = C, P, \tilde{C}, N, W$ the tensor $\nabla_m B_{ijkl}{}^m$ either vanishes or is proportional to the l.h.s. of eq.(33).

Corollary 4.12. *Let $(WZS)_n$ have non singular Z tensor, and be K recurrent with closed λ_i . If $K = C, P, \tilde{C}, N, W$, then ω is a closed 1-form.*

Definition 4.13. A Riemannian manifold is *pseudosymmetric in the sense of R. Deszcz* [17] if the following condition holds:

$$(34) \quad (\nabla_s \nabla_i - \nabla_i \nabla_s) R_{jklm} = L_R (g_{js} R_{iklm} - g_{ji} R_{sklm} + g_{ks} R_{jilm} - g_{ki} R_{jslm} + g_{ls} R_{jkim} - g_{li} R_{jksm} + g_{ms} R_{jkli} - g_{mi} R_{jkl s}),$$

where L_R is a non null scalar function.

In ref.[24] the following theorem is proven:

Theorem 4.14. *In a Riemannian manifold which is pseudosymmetric in the sense of R. Deszcz, it is $R_{im} R_{jkl}^m + R_{jm} R_{kil}^m + R_{km} R_{ijl}^m = 0$.*

Then we can state the following:

Proposition 4.15. *In a $(WZS)_n$ which is pseudosymmetric in the sense of R. Deszcz, if the Z tensor is non-singular then ω_k is a closed 1-form.*

Definition 4.16. A Riemannian manifold is *generalized Ricci pseudosymmetric in the sense of R. Deszcz*, [15], if the following condition holds:

$$(35) \quad (\nabla_s \nabla_i - \nabla_i \nabla_s) R_{jklm} = L_S (R_{js} R_{iklm} - R_{ji} R_{sklm} + R_{ks} R_{jilm} - R_{ki} R_{jslm} + R_{ls} R_{jkim} - R_{li} R_{jksm} + R_{ms} R_{jkli} - R_{mi} R_{jkl s}),$$

where L_S is a non null scalar function.

Theorem 4.17. *In a generalized Ricci pseudosymmetric manifold in the sense of R. Deszcz, it is either $L_S = -\frac{1}{3}$, or $R_{im} R_{jkl}^m + R_{jm} R_{kil}^m + R_{km} R_{ijl}^m = 0$.*

Proof. Equation (35) is transvected with g^{mj} to obtain

$$(\nabla_s \nabla_i - \nabla_i \nabla_s) R_{kl} = L_S [R_{im} (R_{skl}^m + R_{slk}^m) - R_{sm} (R_{ikl}^m + R_{ilk}^m)].$$

Then:

$$\begin{aligned} & (\nabla_i \nabla_k - \nabla_k \nabla_i) R_{jl} + (\nabla_j \nabla_i - \nabla_i \nabla_j) R_{kl} + (\nabla_k \nabla_j - \nabla_j \nabla_k) R_{il} \\ &= 3L_S (R_{im} R_{jkl}^m + R_{jm} R_{kil}^m + R_{km} R_{ijl}^m) \end{aligned}$$

By Lovelock's identity (4.1), the l.h.s. of the previous equation is:

$$\begin{aligned} & \nabla_i \nabla_m R_{jkl}^m + \nabla_j \nabla_m R_{kil}^m + \nabla_k \nabla_m R_{ijl}^m \\ &= (\nabla_i \nabla_k - \nabla_k \nabla_i) R_{jl} + (\nabla_j \nabla_i - \nabla_i \nabla_j) R_{kl} + (\nabla_k \nabla_j - \nabla_j \nabla_k) R_{il} \\ &= -R_{im} R_{jkl}^m - R_{jm} R_{kil}^m - R_{km} R_{ijl}^m. \end{aligned}$$

Compare the two results and conclude that either $L_S = -\frac{1}{3}$, or $R_{im} R_{jkl}^m + R_{jm} R_{kil}^m + R_{km} R_{ijl}^m = 0$. \square

Finally we state:

Proposition 4.18. *In a $(WZS)_n$ which is also a generalized Ricci pseudosymmetric manifold in the sense of R. Deszcz, if the Z tensor is non-singular and $L_S \neq -\frac{1}{3}$, then ω_k is a closed 1-form.*

5. CONFORMALLY HARMONIC $(WZS)_n$: FORM OF THE RICCI TENSOR

In this section we study conformally harmonic $(WZS)_n$ in depth. We show the existence of a proper concircular vector in such manifolds, and obtain the form of the Ricci tensor. The proof only requires the Z tensor to be non singular. For the conformally flat case, in particular, we give the explicit local form of the metric tensor.

The condition $\nabla_m C_{jkl}^m = 0$ is eq.(13) which, by using $R_{ij} = Z_{ij} - g_{ij} \phi$ and the property eq.(9), becomes:

$$(36) \quad \omega_k Z_{jl} - \omega_j Z_{kl} = \frac{1}{2(n-1)} (g_{jl} \nabla_k - g_{kl} \nabla_j) [R + 2(n-1)\phi].$$

This is the starting point for the proofs. By prop 4.7, since Z is non singular, $\omega_k \neq 0$ if and only if $\nabla_k [R + 2(n-1)\phi] \neq 0$.

Remark 5. 1) The condition $\nabla_m C_{jkl}^m = 0$ implies that the manifold is a $(NCS)_n$.
2) If $\nabla_k [R + 2(n-1)\phi] = 0$ the Z tensor is a Codazzi tensor.

The following theorem generalizes a result in [11] for $A(PRS)_n$:

Theorem 5.1. *In a conformally harmonic $(WZS)_n$ the 1-form ω is an eigenvector of the Z tensor.*

Proof. By transvecting eq.(36) with g^{kl} we obtain

$$(37) \quad \omega_j Z - \omega^m Z_{jm} = \frac{1}{2} \nabla_j [R + 2(n-1)\phi];$$

the result is inserted back in eq.(36),

$$\omega_k Z_{jl} - \omega_j Z_{kl} = \frac{1}{(n-1)} [(\omega_k Z - \omega^m Z_{km}) g_{jl} - (\omega_j Z - \omega^m Z_{jm}) g_{kl}],$$

and transvected with $\omega^j \omega^l$ to obtain $\omega_k (\omega^j \omega^l Z_{jl}) = (\omega_j \omega^j) \omega^l Z_{kl}$. The last equation can be rewritten as: $Z_{kl} \omega^l = \zeta \omega_k$ \square

Now eq.(37) simplifies: $\omega_j (\zeta - Z) = -\frac{1}{2} \nabla_j [R + 2(n-1)\phi]$. The result is a natural generalization of a similar one given in ref.[11] for $A(PRS)_n$.

Theorem 5.2. *Let M be a conformally harmonic $(WZS)_n$. Then:*

- 1) M is a quasi Einstein manifold;
- 2) if the Z tensor is non singular and if $(\omega_j \nabla_k - \omega_k \nabla_j) \phi = 0$, then:

$$(38) \quad (\omega_j \nabla_k - \omega_k \nabla_j) \left[\frac{n\zeta - Z}{n-1} \right] = 0,$$

and M admits a proper concircular vector.

Proof. Eq.(36) is transvected with ω^j and theorem 5.1 is used to show that

$$R_{kl} = \left[\frac{Z - \zeta}{n-1} - \phi \right] g_{kl} + \left[\frac{n\zeta - Z}{n-1} \right] \frac{\omega_k \omega_l}{\omega_j \omega^j},$$

i.e. R_{kl} has the structure $\alpha g_{kl} + \beta T_k T_l$ and the manifold is quasi Einstein [7]. By transvecting eq.(23) with g^{jl} we obtain

$$\frac{1}{2} \nabla_k Z + \frac{n-2}{2} \nabla_k \phi = \omega_k Z - \omega^l Z_{kl}.$$

This and theorem (5.1) imply:

$$(39) \quad \frac{1}{2} \nabla_k Z + \frac{n-2}{2} \nabla_k \phi = \omega_k (Z - \zeta).$$

A covariant derivative gives $\frac{1}{2} \nabla_j \nabla_k Z + \frac{n-2}{2} \nabla_j \nabla_k \phi = \nabla_j [\omega_k (Z - \zeta)]$. Subtract the equation with indices k and j exchanged:

$$(Z - \zeta)(\nabla_j \omega_k - \nabla_k \omega_j) + (\omega_k \nabla_j - \omega_j \nabla_k)(Z - \zeta) = 0.$$

According to corollary 4.9, in a conformally harmonic $(WZS)_n$ with non singular Z the 1-form ω_k is closed. Then

$$(40) \quad (\omega_k \nabla_j - \omega_j \nabla_k)(Z - \zeta) = 0$$

Multiply eq.(39) by ω_j and subtract from it the equation with indices k and j exchanged: $(\omega_j \nabla_k - \omega_k \nabla_j)Z + (n-2)(\omega_j \nabla_k - \omega_k \nabla_j)\phi = 0$. Suppose that ω_k , besides being a closed 1-form, has the property $(\omega_j \nabla_k - \omega_k \nabla_j)\phi = 0$, then one obtains the further equation:

$$(41) \quad (\omega_k \nabla_j - \omega_j \nabla_k)Z = 0.$$

Eqs.(40,41) imply the assertion eq.(38). The existence of a proper concircular vector follows from Theorem 3.1. \square

Let us specialize to the case $C_{ijk}^m = 0$ (conformally flat $(WZS)_n$).

It is well known [1] that if a conformally flat space admits a proper concircular vector, then the space is subprojective in the sense of Kagan.

From theorem 5.2 we state the following:

Theorem 5.3. *Let $(WZS)_n$ ($n > 3$) be conformally flat with nonsingular Z tensor and $(\omega_j \nabla_k - \omega_k \nabla_j)\phi = 0$, then the manifold is a subprojective space.*

In [33] K. Yano proved that a necessary and sufficient condition for a Riemannian manifold to admit a concircular vector, is that there is a coordinate system in which the first fundamental form may be written as:

$$(42) \quad ds^2 = (dx^1)^2 + e^{q(x^1)} g_{\alpha\beta}^*(x^2, \dots, x^n) dx^\alpha dx^\beta,$$

where $\alpha, \beta = 2, \dots, n$. Since a conformally flat $(WZS)_n$ with non singular Z tensor admits a proper concircular vector field, this space is the warped product $1 \times e^q M^*$, where (M^*, g^*) is a $(n-1)$ -dimensional Riemannian manifold. Gebarosky [19] proved that the warped product $1 \times e^q M^*$ has the metric structure (42) if and only if M^* is Einstein. Thus the following theorem holds:

Theorem 5.4. *Let M be a n dimensional conformally flat $(WZS)_n$ ($n > 3$). If Z_{kl} is non singular and $(\omega_j \nabla_k - \omega_k \nabla_j)\phi = 0$, then M is the warped product $1 \times e^q M^*$, where M^* is Einstein.*

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